

NORMA PRESMEG

CREATIVITY, MATHEMATIZING, AND DIDACTIZING¹:
LEEN STREEFLAND'S WORK CONTINUES

ABSTRACT. This reaction to the papers in this PME Special Issue of *Educational Studies in Mathematics* draws a wider perspective on the issues addressed and some of the constructs used in research in Realistic Mathematics Education (RME). In particular, it tries to show that while the problems addressed existed within the world-wide arena of mathematics education and were not unique to the Dutch educational system, the methods used at the Freudenthal Institute to address them were uniquely adapted to that system yet foreshadowed developments in the wider field of mathematics education. The predictive aspects of mathematizing, didactizing, and guided reinvention, in which *models-of* become *models-for* on various levels, resonate with trends in mathematics education in recent years, including those promoted by the National Council of Teachers of Mathematics in the USA. Research methodologies, too, have broadened to include more humanistic qualitative methods. Developmental research as epitomized in the RME tradition makes the distinction between quantitative and qualitative research obsolete, because there is no restriction on research methods that may be useful in investigating how to improve the teaching and learning of mathematics, and in the designing of mathematics curricula. Thus some aspects of this research resonate with what have come to be known as multitiered teaching experiments. However, in RME there is also a special content-oriented didactical approach that harmonizes with an emphasis on *didactics* (rather than pedagogy) in several other European countries. Some implications are drawn for future research directions.

KEY WORDS: creativity, didactizing, guided reinvention, humanistic research, mathematizing

Van den Heuvel-Panhuizen (this Special Issue) reminded us that it was Freudenthal's (1977) view that mathematics – in order to be of human value – must be connected to reality. In my own research on how teachers may use the cultural knowledge of learners in their teaching, I have long endorsed both the need for such connections, and the broad interpretation of 'realistic' that includes all that is imaginable, whether related to the material environment or to mathematical abstractions (Presmeg, 1997). When it was suggested that I, as one who has not been involved in *Realistic Mathematics Education Research*, should write a reaction to the papers in this Special Issue honoring Leen Streefland and his work, I hesitated because the ideas that underlie RME and its research resonated so well with my own research endeavors to find ways of linking the lives and lived experiences of learners with formal school mathematics, that I did not know if I could write as one who is outside the enterprise. However,



this resonance encouraged me to accept the challenge of characterizing the valuable contributions of Leen Streefland. When he attended my research report, ‘Cultural mathematics education resources in a graduate course’ at the 18th Annual Meeting of PME in Lisbon (Presmeg, 1994), he remarked to me afterwards that we were on the same journey – and facing the same problems in our work (Presmeg, 1998).

What I attempt to do in this paper is to provide a wider context in which to examine what research in RME has accomplished and what implications arise from Leen Streefland’s work and his legacy, to enliven the debate (sometimes as a devil’s advocate) and to suggest some ideas for further research. I shall not attempt to capture the richness of the individual papers in a few pages (an impossible task); rather I shall choose a few strands that seem to interweave in the papers.

A WIDER CONTEXT

A humanistic view of mathematics education and its research

What comes through clearly in Dörfler’s introduction (this Special Issue) is the humanism of Leen Streefland’s stance towards all aspects of the mathematics education endeavor. In this respect, he is a genuine successor of Freudenthal (1977) with his view of *mathematics as a human activity*. Streefland extended this valuing of the human explicitly – towards learners, teachers, authors of ESM papers, researchers, and towards the nature of the activities in which these people were engaged. *People* is the operative word, and Streefland treated them all with respect, from the learners who were told that they were mathematics researchers (with Streefland and the teacher acting as senior researchers – Elbers, this issue), to the writers who submitted manuscripts and worked with Streefland until they were publishable in ESM. The humanistic view of mathematics education and the people involved in it is not new, but it has gained momentum in the last few decades, manifesting itself in such forms as the realization that learning *mathematics in school is a social activity*, and *consequently that theoretical fields coming from sociology, social and cultural psychology, anthropology, and related human sciences, were legitimate and useful frames* for mathematics education research, complementing those of clinical psychology (Sierpiska and Kilpatrick, 1998). It seems no accident that Alan Bishop’s (1985) keynote address in which he spelled out the social and value-laden nature of the learning of mathematics, took place at Noordwijkerhout in The Netherlands, at the 9th Annual Meeting of PME, with proceedings edited by Streefland. The movement has come so far that there

is currently debate about changing the name of PME, because the psychological aspects are no longer dominant in the mathematics education research reported at meetings of this group.

In addition to the social and value-laden nature of the teaching and learning of mathematics, Streefland's approach embraced a mathematics content-related aspect (elaborated in the next section) that is still an important element of the work of Van den Heuvel-Panhuizen and others at the Freudenthal Institute who are continuing and extending his research (see Van den Heuvel-Panhuizen, 2003).

The content-related didactical approach

There is no mathematics without mathematizing. (Freudenthal, 1973, p. 134)

There is also no mathematics education without mathematics. It follows that in fostering learners' own mathematizing in school, analysis of mathematical content needs to play a central role. In his research into improving the classroom teaching and learning of mathematics, Streefland's emphasis on mathematical content through "didactical phenomenological analysis" (Freudenthal, 1983) of learning in a particular mathematical content area is very recognizable. This approach is evident not only in his own paper in this Special Issue, but also in those of each of the other contributors who describe research that is based on and extends his work. Then it is not surprising that *didactizing*, "the activity of bringing forth didactics" (Yackel et al., this issue), is the obverse side of the coin of *mathematizing* in this research. Both may be *horizontal or vertical*, and both involve processes in which a *model-of* becomes a *model-for* (Van den Heuvel-Panhuizen, this issue; Yackel et al., this issue).

This content-related didactical approach harmonizes well with the *didactics of mathematics* that has special meanings and is a field of study in several European countries, contrasting with the word *pedagogy* that is more often used in connection with the teaching of mathematics in the USA, which lacks this specific emphasis on a piece of mathematical content. Whether it be the learning of percentages or differential equations (two content areas addressed in papers in this issue) the 'What?' of mathematical teaching and learning is of central importance (Van den Heuvel-Panhuizen, 2003) and mathematics as a school subject is addressed in research using this approach as a foundation for the study of teaching and learning. Many classroom aspects may pertain to such research: small-group work and discourse analysis are just two related topics. The *percentage trajectory* (Van den Heuvel-Panhuizen, this issue) illustrates how a *chain of models* is developed in order to guide the teaching and support learners' developing understanding, as both a means and a function of

this growth. This chain of models *enables* the discourse. Apart from knowledge of children and their educational environment, the development of the chain of models requires an in-depth analysis of the mathematical content of percentages, including their relationship with other sub-domains. Thus a domain-specific conceptual framework is developed that may inform the decision making of teachers. As the current papers illustrate, this is another aspect of Streefland's work that is continuing.

Creativity

The kind of learning process that Streefland (this issue) attributed to Stephen Smale in *seeing a mapping as a flow in a space of one dimension higher is an extremely creative kind of mathematizing*. Streefland quoted Stewart (1989, p. 118) as writing that Smale 'went into the Designer Differential Equation business. The subject has never been the same since.' This is the kind of mathematizing that provides a prototype for the design of the curricula described in the papers in this issue. Later in his paper, Streefland even cited the 'divergent production' terminology of Guilford. Already in 1959, Guilford was working out the 'traits of creativity' that later became a part of his Structure-of-the-Intellect model, and in the 1970s he wrote papers with the titles 'Way beyond the I.Q.' and 'Some incubated thoughts on incubation.' With Paul Torrance (1972) writing about teaching children to think creatively, and Edward de Bono (1970) developing the notion of 'lateral thinking' and how to teach this creative mode, all more than three decades ago, one might ask why it took so long for creativity to gain currency in mathematics instructional design. After all, creativity is not new! The 'generative listening' to learners propounded by Rasmussen (Yackel et al., this issue) requires a stance that calls for suspension of preconceived notions. Suspended judgment is one of the requirements in brainstorming as a technique of lateral thinking (De Bono, 1970). Are the current trends (as reflected in the papers in this issue) just a revival of ideas that have been in the literature for more than 30 years?

My answer to these questions is a resounding 'no'. The creativity literature of the 1960s and 1970s was largely in the field of psychology, and while this literature was not ignored in mathematics education at that time (e.g., Presmeg, 1980), *the creativity advocated in current instructional design has a far more sociocultural and didactical-phenomenological basis*. Uncertainty may create room for creativity (Streefland and Van den Heuvel-Panhuizen, 1999). However, *the issue of how to foster creative mathematical thinking in the social milieu of the classroom does raise some interesting challenges*. Elbers (this issue) writes about the dilemma of the teacher's double role in such classrooms. On the one hand, the

teacher is in charge and responsible for the students' activities, planning beforehand, and making immediate pedagogical decisions during the lesson. On the other hand, the teacher 'does not want to frustrate children's creativity by using [his or her] authority by supporting certain answers instead of others.' To overcome the dilemma of this double role, Elbers suggests that Streefland and the co-teacher used three strategies: they encouraged the learners not to be satisfied with one solution, but to search for others; the teachers made global and general suggestions that could be elaborated creatively by learners; and finally, the teachers selected the learners who were asked to share their work with the whole class. In this way the teachers could direct the discussion and at the same time value the children's own constructions. These three strategies appear to be content-neutral, but it was Streefland's recognition of the mathematical content in the students' own work, and careful, forward-looking didactical content analysis, that led to successful mathematizing in this process (both Elbers' and Streefland's papers, and others' in this issue).

This type of pedagogy leads to form-function shifts in learners' constructions that also have echoes in literature from other fields.

Form-function shifts

Taking a learning trajectory on percentage, designed for children in the middle grades, as an example of how didactical models are used in RME to evoke and guide the students' growth in understanding of this topic, Van den Heuvel-Panhuizen (this issue) traces the ways in which a bar model changes its form and its function in the thinking of learners as the unit progresses:

During this process of growing understanding of percentage, the bar gradually changes from a concrete context-connected representation to a more abstract representational model that moreover is going to function as an estimation model, and to a model that guides the students in making the calculations that have to be made.

Allied to these changes in function, the form of the model also changes in the learners' activities, eventually being reduced to a flexible double number line that is useful in other contexts too.

In a different empirical and educational setting, as university students engage with the Stacking Cubes instructional sequence used by Underwood (Yackel et al., this issue), there are also clear form-function shifts in the learners' construction and use of formulas in analyzing linear relationships in patterns of stacked cubes. As Yackel et al. point out, these form-function shifts are reminiscent of those described by Saxe (1991). In his ethnographic study, the form and function of their models of Brazilian

currency changed as the young candy-sellers became inducted into the practice.

With this resonance in previous literature, what is unique in the didactical and developmental research described in this issue is the way in which observed form-function shifts have been used by the researchers in constructing learning trajectories that support instructional design experiments. Anthropologists such as Saxe are not aiming to make any changes in the cultures they study in ethnography. In contrast, a goal of developmental research is to change the culture of the classroom, including the discourse that takes place, during horizontal and vertical mathematizing. Underwood (Yackel et al., this issue) illustrates how proposed chains of signification can be useful planning devices of semiotic evolution in instructional design. After-images of the shifts observed in learners' thinking become pre-images for a learning trajectory that is planned and outlined in a proposed chain of signification.

Argumentation and discourse

Following on from the work of Underwood, Stephan uses Toulmin's model of argumentation (including data, claim, warrant, and backing as constituent parts) in analyzing the structure and functions of students' verbal mathematical contributions during the Stacking Cubes sequence (Yackel et al., this issue). The vertical didactizing that underlies this account harmonizes with and extends the theoretical framework of research in a lesson with middle grades students co-taught by Streefland as a 'community of inquiry' (Elbers, this issue). Whether or not one considers *all* learning of mathematics to be discourse-based (Sfard, 2000; Dörfler, 2000), analysis of discourse – including genres of speech – and the inherent argumentation, in mathematics classrooms, is an important area in which there is growing literature and interest. This significant field of research was also foreshadowed in the work of Leen Streefland and his colleagues, and continued and developed in the research described by Yackel et al. (this issue). As hinted earlier, of particular value in this approach is the role of content-related didactical models in both horizontal and vertical didactizing, undergirding this research and providing a firm theoretical foundation for empirical studies of argumentation and discourse. This is an area of tremendous potential for future research: one gains the impression that what has been accomplished is just a beginning.

Multitiered teaching experiments

The kinds of classroom investigations of mathematics learning described by researchers in this issue (Van Amerom, Elbers, Yackel et al.) are characterized by close cooperation between researchers and teachers in their classrooms, *learners actively engaged in modeling tasks chosen to be experientially real* (in the wide sense that includes vertical as well as horizontal mathematizing – see Van den Heuvel-Panhuizen, this issue), and cycles of development involving curriculum planning, implementation, reflection, and reiteration informed by the previous cycle. In these cycles of developmental research (Gravemeijer, 1998), *working for a prototypical mathematics course at any level of learning from elementary school to college may involve both horizontal and vertical didactizing* (as illustrated by Yackel et al., this issue), as theory on several levels is reflexively put to the test in implementation (in horizontal didactizing), reflected upon and modified (in vertical didactizing). This developmental research is a special form of *multitiered teaching experiment* (Lesh and Kelly, 2000) in which *researchers, teachers, and pupils are all learning in their respective tiers*. Again, what makes this form of teaching experiment powerful is the strong theoretical basis for decision-making in the instructional design heuristics of guided reinvention and *self-generated models (that is, generated by the learners themselves)*. The dynamic nature of this foundation as didactizing allows the theories to morph into new forms as needed, informed by the design team's reflection on the cycles of instructional design. An example of this process is provided in Rasmussen's 'generative listening' (Yackel et al., this issue), which was developed in vertical didactizing with enrichment from the theorizing of other researchers (e.g., Davis in this case). The flexibility provided in this process is suited to the complexity of the endeavor.

One can see a weaving together of threads in the papers in this issue. Streefland's paper describes a creative shift from a post-image to a pre-image in the mathematical thinking of Stephen Smale. This move from descriptive to prescriptive thinking prefigured the school learners' shifts from *models-of* to *models-for* in actively engaging in the activities he described. Another example is given in Van den Heuvel-Panhuizen (this issue). But the shift from post-image to pre-image also characterizes the kind of research that underlies all the curriculum development projects described in this issue. There is unity between mathematizing and didactizing in these multitiered teaching experiments: the school learners' mathematizing has its counterpart in the teachers' and researchers' didactizing, and it is the shift from descriptive to prescriptive elements that is common to all tiers of learning in this theoretical field.

The research is ongoing. Only preliminary assessment has been completed of the implementation of the *Mathematics in Context* middle grades curriculum developed in the USA in close cooperation with researchers from the Freudenthal Institute (Romberg and Shafer, 2003), some units of which were described by Van den Heuvel-Panhuizen (this issue). Illustrating the viability of these research principles in investigating the learning of differential equations and other topics at university level, the research described by Yackel, Stephan, Rasmussen and Underwood (this issue) is also ongoing. Ongoing, too, is the research at the Freudenthal Institute, where Leen Streefland's legacy is most directly felt. Indeed, Leen Streefland's work continues.

SOME IDEAS FOR FURTHER RESEARCH

A plea for reducing the ever-widening gap between researchers and practitioners in mathematics education was made by Mogens Niss (2000) in his plenary lecture at the 9th International Congress on Mathematics Education. . . . In summary, Niss claimed that researchers are not addressing issues that focus on *shaping practice*; rather their issues focus on practice as an *object* of research.

(English, 2002, p. 7, her emphasis)

The *content-related didactical approach* and the *design experiments* described in this issue are squarely in the arena of *shaping practice*. The unity between researchers and practitioners is their strength. However, with its strong emphasis on *using existing theory, as in horizontal didactizing*, and in *furthering theory development*, as in *vertical didactizing*, practice as an object of research in RME is also a significant component. Thus there is also unity in this approach between the two forms of research that Mogens Niss characterized as 'descriptive-analytic' – aiming for understanding of practice, *what is*, as an object of research – and 'normative', in which the question of *what should be* is addressed. The relationship between these two aspects of research is reflexive, each informing the other in cycles of development. While maintaining this unity, there are a number of areas in which one can see current research interests developing, as I have already hinted in the foregoing. The following are issues or questions that come to mind immediately.

- What strategies help teachers to handle the complexities of their multiple and sometimes conflicting roles in teaching for creativity in the way that Streefland advocated and exemplified?
- There is a need to investigate more deeply the reflexive interplay between individual learning and group processes in mathematics classrooms.



- How may learners take ownership by selecting the starting points and contexts themselves in horizontal and vertical mathematizing, and is such ownership fruitful in their learning?
- How do speech genres change in the processes of horizontal and vertical mathematizing in classrooms?
- There is a need to continue the fruitful start of research on argumentation, how it is characterized, how it may be fostered, and its relation to the appreciation of the need for formal proof.

Leen Streefland and his colleagues laid a solid foundation for research on these and other issues. Indeed, his work continues.

ACKNOWLEDGEMENTS

I hereby thank Marja van den Heuvel-Panhuizen and Anna Sierpiska for their incisive and helpful comments on an earlier draft of this paper.

NOTE

1. For consistency with 'mathematizing', this spelling of 'didactizing' is used in this paper.

REFERENCES

- Bishop, A.J.: 1985, 'The social psychology of mathematics education', in L. Streefland (ed.), *Proceedings of the Ninth International Conference for the Psychology of Mathematics Education*, OW&OC, Utrecht University, Utrecht, Vol. II, pp. 1–13.
- De Bono, E.: 1970, *Lateral Thinking: A Textbook of Creativity*, Pelican, London.
- Dörfler, W.: 2000, 'Means for meaning', in P. Cobb, E. Yackel and K. McClain (eds.), *Symbolizing and Communicating in Mathematics Classrooms: Perspectives on Discourse, Tools, and Instructional Design*, Lawrence Erlbaum Associates, Mahwah, New Jersey, pp. 99–131.
- English, L.D.: 2002, 'Priority themes and issues in international research in mathematics education', in L.D. English (ed.), *Handbook of International Research in Mathematics Education*, Lawrence Erlbaum Associates, Mahwah, New Jersey, pp. 3–15.
- Freudenthal, H.: 1973, *Mathematics as an Educational Task*, Reidel Publishing Company, Dordrecht.
- Freudenthal, H.: 1977, 'Antwoord door Prof. Dr H. Freudenthal na het verlenen van het eredoctoraat' [Answer by Prof. Dr H. Freudenthal upon being granted an honorary doctorate], *Euclides* 52, 336–338.
- Freudenthal, H.: 1983, *Didactical Phenomenology of Mathematical Structures*, Reidel Publishing Company, Dordrecht.

- Gravemeijer, K.: 1994, 'Educational development and developmental research in mathematics education', *Journal for Research in Mathematics Education* 25(5), 443–471.
- Gravemeijer, K.: 1998, 'Developmental research as a research method', in A. Sierpiska and J. Kilpatrick (eds.), *Mathematics Education as a Research Domain: A Search for Identity*, Kluwer Academic Publishers, Dordrecht, pp. 277–295.
- Guilford, J.P.: 1959, 'Traits of creativity', in P.E. Vernon (ed.), *Creativity*, Penguin, London.
- Guilford, J.P.: 1977, *Way Beyond the I.Q.*, Creative Education Foundation, Buffalo, New York.
- Guilford, J.P.: 1979, 'Some incubated thoughts on incubation', *The Journal of Creative Behavior* 13(1), 1–8.
- Lesh, R.A. and Kelly, A.E.: 2002, 'Multitiered teaching experiments', in A.E. Kelly and R.A. Lesh (eds.), *Handbook of Research Design in Mathematics and Science Education*, Lawrence Erlbaum Associates, Mahwah, New Jersey, pp. 197–230.
- Presmeg, N.C.: 1980, *Albert Einstein's Thought, Creativity, and Mathematical Education*, Unpublished M. Ed. Thesis, University of Natal.
- Presmeg, N.C.: 1994, 'Cultural mathematics education resources in a graduate course', in J.P. da Ponte and J.F. Matos (eds.), *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education*, University of Lisbon, Lisbon, Vol. 4, pp. 65–72.
- Presmeg, N.C.: 1997, 'A semiotic framework for linking cultural practice and classroom mathematics', in J.A. Dossey, J.O. Swafford, M. Parmantie and A.E. Dossey (eds.), *Proceedings of the Nineteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, ERIC Clearinghouse for Science, Mathematics, and Environmental Education, Columbus, Ohio, Vol. 1, pp. 151–156.
- Presmeg, N.C.: 1998, 'Ethnomathematics in teacher education', *Journal of Mathematics Teacher Education* 1(3), 317–339.
- Romberg, T.A. and Shafer, M.C.: 2003, 'Mathematics in Context (MiC) – preliminary evidence', in S.L. Senk and D.R. Thompson (eds.), *Standards Based School Mathematics Curricula: What Are They? What Do Students Learn?*, Lawrence Erlbaum Associates, Mahwah, New Jersey, pp. 225–250.
- Saxe, G.: 1991, *Culture and Cognitive Development: Studies in Mathematical Understanding*, Lawrence Erlbaum Associates, Mahwah, New Jersey.
- Sfard, A.: 2000, 'Symbolizing mathematical reality into being – or how mathematical discourse and mathematical objects create each other', in P. Cobb, E. Yackel and K. McClain (eds.), *Symbolizing and Communicating in Mathematics Classrooms: Perspectives on Discourse, Tools, and Instructional Design*, Lawrence Erlbaum Associates, Mahwah, New Jersey, pp. 37–98.
- Sierpiska, A. and Kilpatrick, J.: 1998, *Mathematics Education as a Research Domain: A Search for Identity*, Kluwer Academic Publishers, Dordrecht.
- Streefland, L. (ed.): 1985, *Proceedings of the Ninth International Conference for the Psychology of Mathematics Education*, OW&OC, Utrecht University, Utrecht, Vol. I & II.
- Streefland, L. and Van den Heuvel-Panhuizen, M.: 1999, 'Uncertainty, a metaphor for mathematics education?' *The Journal of Mathematical Behavior* 17(4), 393–397.
- Torrance, E. P.: 1972, 'Can we teach children to think creatively?' *The Journal of Creative Behavior* 6(2), 114–143.

Van den Heuvel-Panhuizen, M.: 2003, 'Can research answer the "what" question of mathematics education?' Paper presented at the 10th Conference of EARLI in the Symposium *Provocations to/in Mathematics Education*, August 26–30, 2003, Padova, Italy.

*Mathematics Department,
Illinois State University,
Normal IL 61790-4520, USA,
Telephone +1-309-438 7990, Fax +1-309-438 5866
E-mail: npresmeg@ilstu.edu*

Copyright of Educational Studies in Mathematics is the property of Kluwer Academic Publishing and its content may not be copied or emailed to multiple sites or posted to a listserv without the copyright holder's express written permission. However, users may print, download, or email articles for individual use.

Copyright of Educational Studies in Mathematics is the property of Springer Science & Business Media B.V. and its content may not be copied or emailed to multiple sites or posted to a listserv without the copyright holder's express written permission. However, users may print, download, or email articles for individual use.